Algebra. TST preparation

1. Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ which satisfy the equation:

$$f(x)f(y) = 2f(x + yf(x))$$

for all $x, y \in \mathbb{R}^+$.

2. Find all the functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$

3. Determine all pairs of functions (f, g) from the set of real numbers to itself that satisfy:

$$g(f(x + y)) = f(x) + (2x + y)g(y)$$

for all $x, y \in \mathbb{R}$

4. Let $\mathbb{Q}^+$ be the set of positive rational numbers. Construct a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that

$$f(xf(y)) = \frac{f(x)}{y}$$

for all $x, y \in \mathbb{Q}^+$

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(x^2 + f(y)\right) = y + (f(x))^2 .$$

for all $x, y \in \mathbb{R}$

6. Let f and g be two functions from the set of positive integers to itself. Suppose that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integers. Prove that $f(n) = g(n)$ for any positive integer n.

7. Suppose that $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers which satisfy the inequality

$$(a_1^2 + a_2^2 + \cdots + a_n^2 - 1)(b_1^2 + b_2^2 + \cdots + b_n^2 - 1) > (a_1b_1 + a_2b_2 + \cdots + a_nb_n - 1)^2.$$

Prove that $a_1^2 + a_2^2 + \cdots + a_n^2 > 1$ and $b_1^2 + b_2^2 + \cdots + b_n^2 > 1$.

8. Let x, y and z be positive real numbers with $xyz = 1$. Prove that the following inequality holds:

$$\sqrt[3]{\frac{x+y}{2z}} + \sqrt[3]{\frac{x+z}{2y}} + \sqrt[3]{\frac{z+y}{2x}} \le \frac{5(x+y+z)+9}{8}$$

9. Let $x_1, \dots, x_{100}$ be nonnegative real numbers such that $x_i + x_{i+1} + x_{i+2} \le 1$ for all $i = 1, \dots, 100$ (we put $x_{101} = x_1, x_{102} = x_2$.). Find the maximal possible value of the sum $S = \sum_{i=1}^{100} x_i x_{i+2}$.

10. Prove that in any set of 2000 distinct real numbers there exist two pairs $a > b$ and $c > d$ with $a \ne c$ or $b \ne d$, such that

$$\left| \frac{a-b}{c-d} - 1 \right| < \frac{1}{100000}.$$

11. Let n be a positive integer and $x_1, x_2, \dots, x_n$ be some fixed positive reals. Prove that there exist numbers $a_1, a_2, \dots, a_n \in \{-1, 1\}$ such that the following inequality holds:

$$a_1 x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2 \ge (a_1 x_1 + a_2 x_2 + \cdots + a_n x_n)^2$$

12. Let $x_1, x_2, \dots, x_n$ be real numbers satisfying the conditions:

$$\begin{cases} |x_1 + x_2 + \cdots + x_n| & = & 1 \\ |x_i| & \le & \dfrac{n+1}{2} \qquad \text{for } i = 1, 2, \dots, n. \end{cases}$$

Show that one can find a permutation $y_1, y_2, \dots, y_n$ of $x_1, x_2, \dots, x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \le \frac{n+1}{2}.$$

13. Consider a sequence $a_1, a_2, \cdots, a_n$ of positive integers. Put $a_{n+i} = a_i$ for all $i \ge 1$. If

$$a_1 \le a_2 \le \cdots \le a_n \le a_1 + n$$

and

$$a_{a_i} \le n + i - 1 \quad \text{for} \quad i = 1, 2, \cdots, n,$$

prove that

$$a_1 + \cdots + a_n \le n^2.$$

14. Let $a_1, a_2, \dots$ be an infinite sequence of positive real numbers for which there exists a real number c with $0 \le a_i \le c$. Suppose that

$$|a_i - a_j| \ge \frac{1}{i+j} \quad \text{for all } i,\ j \text{ with } i \ne j.$$

Prove that $c \ge 1$

15. Assume that a sequence of positive real numbers $a_1, a_2, \ldots$ satisfies the relation $a_n = |a_{n+1} - a_{n+2}|$ for all $n \geq 0$, and $a_0 \neq a_1$. Can this sequence be bounded?

16. Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1},\ s_{s_2},\ s_{s_3},\ \ldots \qquad \text{and} \qquad s_{s_1+1},\ s_{s_2+1},\ s_{s_3+1},\ \ldots$$

are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

17. Two sequences of integers $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$, satisfy the equation

$$(a_n - a_{n-1})(a_n - a_{n-2}) + (b_n - b_{n-1})(b_n - b_{n-2}) = 0$$

for each integer greater than 2. Prove that there exists a positive integer k such that $a_k = a_{k+2016}$

18. Suppose that α is a root of a polynomial with integral coefficients. Show that there exist two constants $C > 0$ and $k \in \mathbb{N}$ such that for any rational number $\frac{p}{q}$ the inequality $|\alpha - \frac{p}{q}| > \frac{C}{q^k}$ holds.